Nonuniversal quantities from dual renormalization group transformations

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Using a simplified version of the renormalization group (RG) transformation of Dyson's hierarchical model, we show that one can calculate all the nonuniversal quantities entering into the scaling laws by combining an expansion about the high-temperature fixed point with a dual expansion about the critical point. The magnetic susceptibility is expressed in terms of two dual quantities transforming covariantly under an RG transformation and has a smooth behavior in the high-temperature limit. Using the analogy with Hamiltonian mechanics, the simplified example discussed here is similar to the anharmonic oscillator, while more realistic examples can be thought of as coupled oscillators, allowing resonance phenomena. [S1063-651X(99)05209-5]

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One important contribution of the renormalization group (RG) method is to show that there exists a close connection [1] between statistical mechanics near criticality and Euclidean field theory in the large-cutoff limit. In this approach, the determination of the renormalized quantities at zero momentum amounts to the determination of a certain number of parameters appearing in the scaling laws. Some of these parameters are universal (the critical exponents) and much effort has been successfully devoted to their calculation. On the other hand, new techniques need to be developed in order to reliably calculate the nonuniversal parameters.

We limit here the discussion to the case of scalar field theories with a lattice regularization (spin models). This class of models has several important applications in particle physics (e.g., self-interactions in the Higgs sector) and condensed-matter physics (e.g., ferromagnetism) which require an accurate nonperturbative treatment. For β, the inverse temperature (or the hopping parameter), close to its critical value β_c, one can express the magnetic susceptibility (zero-momentum two-point function) with an expression which, when $D<4$, takes the form [2]

$$\chi \simeq (\beta_c-\beta)^{-\gamma}[A_0+A_1(\beta_c-\beta)^{\Delta}+\cdots], \tag{1}$$

where the nonuniversal quantities $A_0, A_1, \ldots$ are functions of the other (''bare'') parameters of the theory. Following the discussions of Refs. [1,3], we can use Eq. (1) to obtain a nonperturbative definition of the renormalized mass m_R^2 of the form

$$m_R^2=\frac{\Lambda_R^2}{A_0+A_1(\Lambda_R/\Lambda)^{2\Delta/\gamma}+\ldots}, \tag{2}$$

for a scale of reference Λ_R, and a uv cutoff Λ. Similar considerations apply to the other renormalized quantities which can be obtained from the higher point functions. In order to complete in a quantitative way this nonperturbative renormalization program, one needs to be able to calculate the nonuniversal quantities in Eq. (2) as well as the universal ones.

This task can be achieved [3] in the case of the well-studied hierarchical model [4–7]. Using the numerical methods developed in Ref. [7], one can calculate the susceptibility at various values of β and extract the unknown parameters in Eq. (1) by direct fits [3]. This is a rather tedious procedure involving successive numerical refinements. A more satisfactory approach consists in expanding about the fixed point calculated by Koch and Wittwer [5]. In a system of coordinates where the fixed point is at the origin and the axes coincide with the eigenvectors of the linearized transformation, the RG transformation reads

$$d_{n+1,m}=\lambda_m d_{n,m}+\sum_{k,l}\Gamma_m^{kl}d_{n,k}d_{n,l}, \tag{3}$$

where the λ_m are the eigenvalues of the linearized RG transformation (which yield the critical exponents) and the Γ_m^{kl} are calculable coefficients. In Ref. [3], we found that the direct fits and the linearization agrees with 12 significant digits for the leading exponent γ. The linearization method does not provide a way to calculate the nonuniversal quantities $(A_0, A_1 \ldots)$. It would be a great accomplishment to show that these nonuniversal quantities could be calculated accurately by taking into account the nonlinear terms in Eq. (3). In this paper we show that such a calculation can be performed in a one-variable version of Eq. (3) which is justified in the next paragraph. Furthermore, some of the calculations can be performed much more efficiently by combining the expansion described above with a dual expansion which can be identified with the high-temperature expansion.

In the following, we consider a recursion relation for the magnetic susceptibility which reads

$$\chi_{n+1}=\chi_n+(\beta/4)(c/2)^{n+1}\chi_n^2, \tag{4}$$

where $c=2^{1-2/D}$ in order to approximate a D-dimensional model and n stands for the fact that the susceptibility is calculated with a number of sites 2^n. In the following, we limit ourselves to a range of parameters corresponding to ferromagnetic interactions in the symmetric phase, and such that an infinite volume limit exists. This means $0<\beta<\beta_c$ (the value β_c is calculated below) and $0<c<2$. Equation (4) can be obtained as follows. First, we consider the recursion formula for the hierarchical model in the approximation where the Fourier transform of the local measure is approximated by a polynomial of degree 2 (this is called $l_{\max}=1$ in Ref.

[7]) and then we expand the resulting recursion formula for the susceptibility to first order in β. The variable is then rescaled in order to obtain a recursion formula in terms of the physical quantity χ. The recursion formula Eq. (4) becomes an accurate approximation of the exact recursion formula for Dyson's model when n is large enough. A related formula is used in Ref. [7] to estimate the finite volume effects [see Eq. (5.1) therein]. In the following, we use the notation ξ for $c/2$. For definiteness, we will take the initial value $\chi_0 = 1$.

The explicit dependence on n and β in Eq. (4) can be eliminated by introducing $h_n \equiv \alpha \xi^n \chi_n$. The constant of proportionality α can be fixed by requiring that the fixed points of the the RG transformation in terms of the new variable are 0 and 1. This yields $\alpha = \beta c^2 / [8(2-c)]$. The initial value $h_0 = 1$ (the unstable fixed point), corresponds to the choice $\beta = \beta_c = 8(2-c)/c^2$. In summary

$$h_n = (\beta/\beta_c)\xi^n \chi_n, \tag{5}$$

and the recursion formula then becomes a simple quadratic map (called the ''h map'' hereafter)

$$h_{n+1} = \xi h_n + (1-\xi) h_n^2, \tag{6}$$

together with the initial condition $h_0 = \beta/\beta_c$. The restriction to $0 < \beta < \beta_c$ corresponds to the range $0 < h_0 < 1$ which implies that for positive and finite n, h_n stays within this interval. Note that Eq. (6) can be used to give h_n as a function of h_{n+1}. This quadratic equation has two solutions; however, if we require $0 < h_n, h_{n+1} < 1$, only one solution is acceptable and a unique inverse can be obtained by this restriction. If we impose this restriction, the term ''group'' in RG can be understood in its proper sense.

We now discuss the two fixed points. The fixed point $h_0 = 0$ corresponds to the choice $\beta = 0$ and is called the high-temperature (HT) fixed point. Remembering that $0 < \xi < 1$, we see that the HT fixed point is stable, with eigenvalue ξ in the linear approximation. Using a graphical representation of the quadratic map, one sees that the HT fixed point is globally attractive for the interval (0,1). At the other end, the fixed point $h_0 = 1$ corresponds to the choice $\beta = \beta_c$, and is called the critical point. This fixed point is unstable, with eigenvalue $\lambda = (2-\xi)$. Note that if ξ is fixed by our initial choice of the dimensionality parameter D, the value of λ can be seen as an approximate value for largest eigenvalue λ_1 of the hierarchical model. This value is not too far off numerically. For instance for $D = 3$, $2 - \xi \simeq 1.37$ which can be compared with the known [4,3] value 1.427 17 . . . .

We can expand the h map about the unstable fixed point by using the reparametrization $h_n = 1 - d_n$. Note that $d_0 = (\beta_c - \beta)/\beta_c$ is the variable which appears in the parametrization of the susceptibility given by Eq. (1). The recursion formula for d_n reads

$$d_{n+1} = \lambda d_n + (1-\lambda) d_n^2, \tag{7}$$

with $\lambda = 2 - \xi$. We will call this map the d map. This map can be seen as a one variable version of Eq. (3). Note the similarity with the original h map. One can introduce a duality relation between the two maps which interchanges $h_n \leftrightarrow d_n$ and $\xi \leftrightarrow \lambda$. In the following, we use the notations $h_n = \widetilde{d}_n$ and $\xi = \widetilde{\lambda}$ to express the quantities appearing in the h map as dual to the one appearing in the d map. If the duality transformation is applied twice, one returns to the original quantities.

Recalling that $0 < h_0 < 1$, we also have $0 < d_0 < 1$ with small values (approaching 0 from above) in one variable corresponding to ''large'' values (approaching 1 from below) values in the dual variable. We would like to construct an expression for χ which is accurate for both small and large values of d_0. In order to do this, we need to use Eq. (7) beyond the linear approximation. In the linear approximation, which is justified when d_0 is very small (β close to β_c), $d_n \simeq \lambda^n d_0$. The linear approximation breaks down for values of n of order n^* defined by the relation $\lambda^{n^*} d_0 = 1$. For n larger than n^*, the nonlinear terms become important and d_n approaches 1 from below as dictated by the global attractiveness of the HT fixed point. For n large enough, the linearized h map can be used to show that the HT fixed point is reached exponentially fast. The linearization about the critical point provides the usual type of expression for the critical exponent γ: since $\chi \sim \xi^{-n^*}$,

$$\gamma = -\ln \xi / \ln \lambda. \tag{8}$$

In order to refine the order of magnitude estimate given by the leading singularity, we will express d_n as a function of d_0. For this purpose, we first construct a function $y(d)$ which transforms covariantly under Eq. (7):

$$y[\lambda d + (1-\lambda) d^2] = \lambda y(d). \tag{9}$$

If we add the requirement that for small values d, $y(d) \simeq d$, this equation has a unique solution as a power series in d:

$$y(d) = d + \frac{d^2}{\lambda} + \frac{2d^3}{\lambda(\lambda+1)} + \frac{1+5\lambda^2}{\lambda^2(1+\lambda)(1+\lambda+\lambda^2)} + \cdots. \tag{10}$$

The inverse function can be constructed similarly. In both cases, the coefficients can be calculated by simple recursion relations, easily implementable on a computer. We can now write

$$d_n = y^{-1}(\lambda^n y(d_0)). \tag{11}$$

The idea of using intermediate variables with simple transformation properties has a long history, for instance, the angle-action variables in Hamiltonian mechanics and the normal form of differential equations appearing in Poincaré's dissertation. For continuous RG transformations, Wegner [8] introduced the notion of scaling variables. In the case discussed here, the construction of y can be seen as a discrete version of Wegner's procedure.

Much can be said about the convergence properties of $y(d)$ and its inverse. A numerical analysis of the coefficients indicates very clearly that y^{-1} is an entire function, while y is analytical on the open disk of radius 1 and has a power singularity when its argument tends to 1. Consequently, when $0 < d_0 < 1$, one can always find an accurate expression for d_n by using sufficiently many terms in the expansions of y and y^{-1}. Note that Eq. (11) can also be used at negative values of n, providing the inverse transformations, which can be uniquely defined by requiring—as in the case of the h map discussed above—that the preimage lies in the (0,1) interval. With this requirement, the transformation of Eq. (7) becomes a group and the function $y(d)$ a nonunitary representation of this group. Note that one could also define a continuous transformation by extending n to all the real values.

Everything we have done for the d map can be repeated almost verbatim for the h map. We can construct a dual function $\tilde{y}(\tilde{d})$ transforming covariantly under the h map with an expansion in $\tilde{d}$ of the form of the one given in Eq. (10) but with λ replaced by $\tilde{\lambda}=\xi$. As for $y(d)$, we have a clear numerical indication that $\tilde{y}(\tilde{d})$ is analytical on the open disk of radius 1 and has a power singularity when its argument tends to 1. On the other hand, its inverse is not entire but has a finite radius of convergence with a square-root behavior at the intersection of the boundary of the disk of convergence and the negative real axis. Recalling that $0<\xi<1$ and that $\tilde{d}_0=h_0=\beta/\beta_c$ we see that for n large enough, we can use the above-described expansions to calculate

$$h_n=\tilde{y}^{-1}(\xi^n\tilde{y}(h_0)). \tag{12}$$

In the limit where n becomes infinite, the argument of $\tilde{y}^{-1}$ goes to zero and it is justifiable to retain only the first term of its expansion. Using the definition of the susceptibility of Eq. (5), we find that the ξ dependence cancels and that we obtain the high-temperature expansion

$$\chi\equiv\lim_{n->\infty}\chi_n=\frac{\tilde{y}(h_0)}{h_0}=1+\beta\frac{c}{4(2-c)}+\cdots. \tag{13}$$

This expansion has features which are in qualitative agreement with the actual HT series [6,9] of the hierarchical model.

One can, in principle, use this HT expansion to extract the leading and subleading singularities of χ. However, this procedure is in general very inefficient because small physical effects can be amplified dramatically in this expansion. For instance, for $\lambda=1.8$, the sequence of ratios of successive coefficients is completely ''noisy'' and no information can be extracted from it. When λ is lowered, the ''noise'' decreases and takes the form of smooth log-periodic oscillating terms as in the examples discussed in Ref. [6].

Instead of using the HT expansion, we would like to have an expansion in terms of the dual variable d_0. Such a goal can be achieved by combining the two covariant quantities y and $\tilde{y}$ into one invariant quantity which we call A below. Using the definition of γ given in Eq. (8), one sees that $\lambda^{\gamma}=\tilde{\lambda}^{-1}$ and consequently

$$A=(y(d_n))^{\gamma}\tilde{y}(h_n) \tag{14}$$

is n independent. A can be called a constant of motion or an RG invariant. We can now rewrite

$$\chi=\frac{A}{(1-d_0)(y(d_0))^{\gamma}}. \tag{15}$$

We will show later that A is a bounded function for $0<d_0<1$. Equation (15) makes us suspect that χ has a singularity when d_0 becomes close to 1, or in other words, when β becomes small. On the other hand, we know that in this limit $\chi=1$. This apparent difficulty can be resolved by noticing that $\tilde{y}$ has a singularity with power $-\gamma=\ln\tilde{\lambda}/\ln\lambda$, consequently the dual quantity y has a power singularity with dual exponent: $\tilde{\gamma}=1/\gamma$. Consequently, $(y(d_0))^{\gamma}$ at the denominator cancels the singularity and the expansion extends globally.

We now calculate A expressed as a function of $y(d_0)\equiv y_0$. The invariance of A under a RG transformation implies the discrete scale invariance:

$$A(\lambda y_0)=A(y_0), \tag{16}$$

and the Fourier mode expansion:

$$A(y_0)=\sum_{n=-\infty}^{+\infty}a_n y_0^{in\omega} \tag{17}$$

with $\omega=2\pi/\ln\lambda$, and consequently

$$a_n=\frac{1}{\ln\lambda}\int_{y_a}^{\lambda y_a}dy_0 y_0^{-1-in\omega}A(y_0). \tag{18}$$

The lower value y_a of the integration interval is arbitrary and we can choose it at our convenience and construct a decent series approximation for A which is accurate in the integration interval. More explicitly, we can rewrite

$$A(y_0)=(y_0)^{\gamma}\tilde{y}[1-y^{-1}(y_0)], \tag{19}$$

and use the series expansions for y^{-1} and $\tilde{y}$. If y_0 is small, we need a few terms for y^{-1} and many for $\tilde{y}$. If y_0 is large, we need many terms for y^{-1} and a few terms for $\tilde{y}$. These two extreme possibilities are very inefficient ways to calculate the Fourier coefficients. We have compared the approximate values $a_0(m,\tilde{m})$ obtained from expansions of Eq. (19) with m terms for y^{-1} and $\tilde{m}$ for $\tilde{y}$ with an accurate value of a_0 and found the approximate behavior

$$|a_0(m,\tilde{m})-a_0|\propto\exp[-K_1(m+\tilde{m})+K_2(m-\tilde{m})^2], \tag{20}$$

where K_1 and K_2 are positive constants. Equation (20) implies that for $m+\tilde{m}$ fixed, it is very advantageous to pick the ''self-dual'' option $m\simeq\tilde{m}$.

For values of ξ not too small, the contribution of the nonzero Fourier modes to the susceptibility is exponentially suppressed. We found indications for the following behavior:

$$|a_n|/|a_0|\propto\exp(-|n|\pi\omega/2), \tag{21}$$

as in another example of function with log-periodic oscillations discussed in Sec. V of Ref. [6]. The first indication is the shape of the basin of attraction of the stable fixed point of the h map, in the complex h plane. Near the unstable fixed point ($h=1$, $d=0$), we can linearize $y\simeq d=1-h$. From Fig. 1, we see that if $d=-\delta\exp(i\theta)$ with $0<\delta\ll 1$, the points such that $|\theta|<\pi/2$ are attracted to zero. Given the

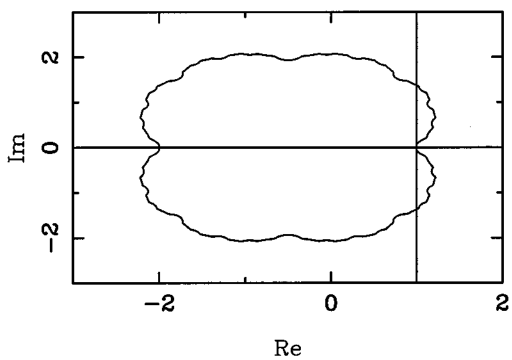

FIG. 1. Boundary of the basin of attraction of 0 for the complex h map with $\xi = 0.5$.

behavior of $y^{in\omega}$ in this region, this requires for large values of n the suppression given by Eq. (21). We have also checked explicitly for $n=1$ and 2, that this exponential suppression provides a good fit of the data for $\omega > 11$. Despite the suppression, the nonzero modes are quite visible in the HT expansion because of a factor $[\Gamma(\gamma + in\omega)]^{-1}$ which appears in the the expression of the HT coefficients [see Eq. (3.7) of Ref. [6]] and cancels, in leading order, the suppression from Eq. (21).

The construction of covariant quantities extends easily to the case of several variables and can be used to attempt accurate calculations with the hierarchical model. The extension to the case of the nearest-neighbor model is a more difficult procedure. It is clear that one can as in Ref. [7] use polynomial approximations for the Fourier transform of the local measure and write a recursion formula for a block-spin transformation. However, this procedure generates nonlocal interactions. It is not clear if these interactions can be exponentiated in a compact way (as in the Gaussian case). The present work should be seen as an encouragement to attack this difficult question.

Two remarks can be made regarding the construction of covariant quantities in the multivariable case. The first one is that for a l-dimensional quadratic recursion formula, the number of coefficients to be determined at order m grows like l^m and consequently optimization is an important consideration. Second, the problem has no solution if a given eigenvalue can be written exactly as a product of other eigenvalues. This is the exponentiated form of the problem of logarithmic anomalies raised by Wegner in Ref. [8]. Generically, such a problem is likely to occur is an approximate way. For instance, if we use the eigenvalues for the $D=3$ hierarchical model given in Ref. [3], we find that $\lambda_3 - \lambda_2^5 \simeq 10^{-2}$. When this is the case, we have a ‘‘small denominator problem’’ which reflects approximate ‘‘resonance’’ among the various ‘‘modes’’ present. Pursuing this analogy, the results presented here provide a solution of a nonlinear problem with one degree of freedom. Their application to realistic systems seems likely to have a complexity and an interest comparable to systems of coupled nonlinear oscillators.

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